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Matched Asymptotic Expansion for Caged Black Holes - Regularization of the Post-Newtonian Order

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Abstract: The "dialogue of multipoles" matched asymptotic expansion for small black holes in the presence of compact dimensions is extended to the Post-Newtonian order for arbitrary dimensions. Divergences are identified and are regularized through the matching constants, a method valid to all orders and known as Hadamard's partie finie. It is closely related to "subtraction of self-interaction" and shows similarities with the regularization of quantum field theories. The black hole's mass and tension (and the "black hole Archimedes effect") are obtained explicitly at this order, and a Newtonian derivation for the leading term in the tension is demonstrated. Implications for the phase diagram are analyzed, finding agreement with numerical results and extrapolation shows hints for Sorkin's critical dimension – a dimension where the transition turns second order.

Contents

1. Introduction

The black-string black-hole transition (which includes the Gregory-Laflamme instability [1]) is a phase transition in General Relativity which occurs at higher dimensions $d > 4$ where black hole uniqueness fails. It is known to raise deep issues including topology change and critical dimensions [2] as well as naked singularities and thunderbolts (see the reviews [3, 4] and references therein).

In order to test these issues, black object solutions were sought both analytically and numerically in $\mathbb{R}^{d-2,1} \times \mathbf{S}^1$. One limit which is amenable to analytic study is the small black hole limit, where ρ_0, the black hole radius, is much smaller than L, the size of the compact dimension. Admittedly, small black holes are far from being the large black holes ($\rho_0 \sim L$) which are involved in the phase transition and the latter should be studied numerically. Yet, they are still very useful in providing the following

- Hints about large black holes via extrapolation.
- Important tests for numerics.

In [5] we introduced a general perturbation theory for small black holes, an implementation of the method of "matched asymptotic expansion" to the static case, which was termed "a dialogue of multipoles" since it describes how field multipoles change the black hole's shape (or mass multipoles) and these in turn change the field and so on. This method requires to obtain a solution in two zones: the near horizon zone and the asymptotic zone, solutions which must agree over the overlap region.

In principle it should be possible to devise a perturbation theory in a single zone. For instance, Harmark wrote an approximate solution in a single zone [6], but nobody extended it to a full perturbation series. However, it is probably simpler to use the two-zone method since it does not involve the arbitrariness of an initial guess, and since the differential operator to be inverted is the same at each order.

In [5] we obtained the leading order corrections to the metric. Recently a "matched asymptotic expansion" was used to obtain the explicit solutions in 5d up to the next to leading order ($\mathcal{O}({\rho_0}^4)$, $\mathcal{O}(L^{-4})$) [7] (see also [8] for a closely related work in the braneworld context).

In this paper we extend our method to the next to leading order in the asymptotic zone, namely to Post-Newtonian order, for arbitrary d (namely, order $2(d-3)$). It turns out that at this order a new qualitative phenomenon appears – *divergences*. Such divergences are familiar from Post-Newtonian studies, but we are not aware of a general prescription that works at all orders. In section 2 we obtain such a clear regularization in terms of a "cut-off and match" method, which relies on Hadamard's "Partie finie" regularization and is summarized in equation (2.15). This method shows some similarity with the regularization of quantum field theories (see [9] for related ideas). Alternatively, a certain subtraction of self-interaction is shown to be an equivalent method at this order. We believe this equivalence should continue to hold at arbitrary orders, but this is not manifest in the current setting. Moreover, we find that at this order the matching is quite trivial and matching constants vanish. Again we do not know whether this can be made to persist to higher order and whether it depends on a clever choice of gauge order by order.

The above-mentioned divergences did not show up in neither [6] nor [7]: [6] was using a different, single patch, method while in [7] the authors succeeded in obtaining full explicit solutions for the metric without the use of Green's function, thereby circumventing the issue of divergences. Still, the regularization of these divergences is important both conceptually and practically: conceptually, they are a general feature of the two-zone method and show up at all high enough orders; and practically, they are essential if one wishes to use Green's function either while resorting to a numerical solutions for the whole metric, or when computing analytically the solutions' asymptotics, namely the thermodynamics. Moreover, similar divergences are very common in Post-Newtonian studies in general.

The pre-existing results for the leading corrections are reviewed in section 3. Proceeding to next to leading order, we met difficulties in trying to obtain explicit expressions for the full metric, but we were able to compute the next to leading corrections to the black hole's mass and tension, using our regularization procedure, thereby confirming the results of [6] with our different method. The computation is described in section 4 and the results are presented in subsection 4.3. The leading behavior of the tension is given an intuitive

Newtonian explanation in subsection 4.4. A higher order expression for the "black hole Archimedes effect" is given in Appendix B.

Finally, we discuss the implications for the phase diagram in section 5, in light of the objectives presented in the beginning of this introduction. Within the small black hole region of validity we find very good agreement with numerical data in 5d and 6d [10, 11, 12]. Extrapolating beyond this region we find evidence for a second order phase transition at large enough dimensions, which is an indication for Sorkin's critical dimension [13].

1.1 Set-up

Figure 1 illustrates the coordinates which we use. We denote by z, r the "cylindrical" coordinates, where z is the coordinate along the compact dimension whose period we denote by L, and r is the radial coordinate in the extended $\mathbb{R}^{d-2}$ spatial dimensions. In addition we introduce "spherical" coordinates ρ, χ. ρ_0 is a characteristic size of the horizon. We are interested in obtaining solutions for small black holes, $\rho_0 \ll L$, so ρ_0/L is the small parameter. The "spherical" coordinates will be more natural in the vicinity of the black hole since it is nearly spherical in the small black-hole limit.

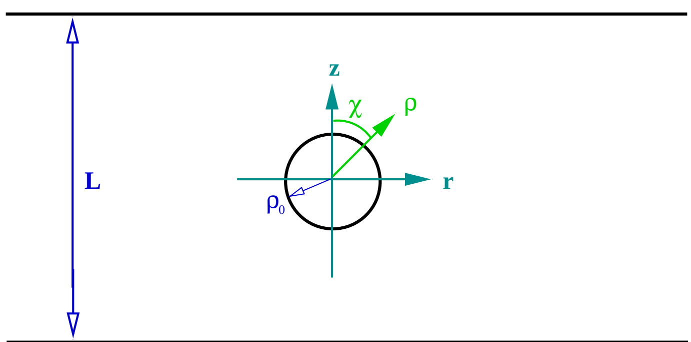

Figure 1: Illustration of the (r, z) "cylindrical" coordinates and the (ρ, χ) "spherical" near horizon coordinates. The period of the compact dimension (in the z direction) is denoted by L and the radius of the black hole is denoted by ρ_0.

The general method of "dialogue of multipoles" matched asymptotic expansion was explained in [5]. The static nature of the problem under study makes this application of matched asymptotic expansion more transparent than the usual 4d applications. In this method we consider two zones (see figure 2):

- The *asymptotic zone* where $\rho \gg \rho_0$ and ρ_0 is the small parameter. The zeroth order solution is flat space with a periodic coordinate $z \sim z + L$ and the point at the origin $(r, z) = (0, 0)$ removed.

- The *near zone* where $\rho \ll L$ and L^{-1} is the small parameter. The zeroth order solution is the Schwarzschild black hole with radius ρ_0.

Thus in the asymptotic zone we expand the metric in ρ_0 and in the near zone we expand

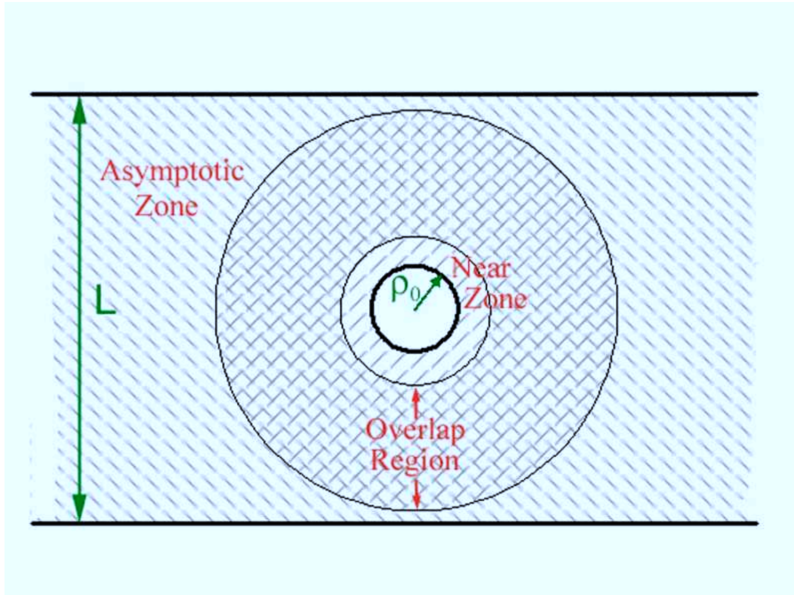

Figure 2: The division of the spacetime into two zones: the **near zone** $\rho \ll L$ where ρ_0 is fixed and the perturbative parameter is L^{-1}, and the **asymptotic zone** $\rho \gg \rho_0$ where L is fixed and the perturbative parameter is ρ_0. The two zones overlap over the **overlap region** which increases indefinitely in the small black hole limit. During the perturbation process the two zones are separate, and communicate only through the matching "dialogue". The near zone is defined by $\{(\rho,\chi):\ \rho \geq \rho_0\}$ while the asymptotic zone is defined by $\{(r,z):\ r \geq 0,\ z \sim z+L\}\backslash(0,0)$.

it in L^{-1}

$$\begin{aligned} g_{\mu\nu}^{(\text{asymp})}(x) &= \sum_{j=0}^{\infty} {\rho_0}^j\, g_{\mu\nu}^{(\text{asymp},j)}(x)\ , \\ g_{\mu\nu}^{(\text{near})}(x) &= \sum_{j=0}^{\infty} L^{-j}\, g_{\mu\nu}^{(\text{near},j)}(x)\ . \end{aligned} \tag{1.1}$$

Orders which are integral multiples of $d-3$ have a special role, and therefore we introduce a special square brackets notation for them

$$g_{\mu\nu}^{[k]} := g_{\mu\nu}^{(k(d-3))}\ . \tag{1.2}$$

The metric in the two regions must be consistent over the overlap region $\rho_0 \ll \rho \ll L$, in a double expansion in ρ_0, L. This is the "matching procedure", whose first steps are summarized in figure 3. For example, the first match is between the Schwarzschild-Tangherlini solution in the near zone (order L^{-0}) and the Newtonian approximation (order ρ_0^{d-3}).

2. Divergences and regularization

Divergences. When we proceed in the "dialogue of multipoles" matched asymptotic expansion beyond the leading order we encounter a new qualitative feature – divergences.

These occur in both the asymptotic zone and the near zone, and for concreteness we shall first describe the divergences and their regularization for the asymptotic zone in detail, and later the near zone will be described briefly.

At each order, k, in the perturbation series we get an equation of the form

$$L\, h^{(k)} = Src^{(k)} \ , \tag{2.1}$$

where L is the linear operator which appears at first order, $h^{(k)}$ is a function which determines the perturbation to the metric at order k and $Src^{(k)}$ is a source term which depends on metric functions from lower orders. More specifically, in the asymptotic zone we have

$$\triangle\, h^{(k)} = Src^{(k)} \sim \partial^{k_1}\, \Phi^{k_2} \ , \tag{2.2}$$

where $\triangle$ is the Laplacian, Φ is the Newtonian potential given explicitly by

$$\Phi := \rho_0^{d-3} \sum_{n=-\infty}^{\infty} \frac{1}{\left(r^2 + (z+nL)^2\right)^{\frac{d-3}{2}}}, \ , \tag{2.3}$$

∂^{k_1} is a symbolic notation for some differential operator of order k_1 and $k_1, k_2 \geq 2$. [1] For small ρ Φ behaves as $\Phi \sim 1/\rho^{d-3}$ and for large r as $\Phi \sim 1/r^{d-4}$. For example the Post-Newtonian equations (PN) are

$$\begin{aligned} \triangle \left(h_{tt}^{[2]} + \frac{1}{2}\,\Phi^2 \right) &= 0 \ , \\ \triangle \left(h_{ij}^{[2]} - \frac{1}{2}\,\delta_{ij} \left(\frac{\Phi}{d-3} \right)^2 \right) &= -\frac{d-2}{2(d-3)}\, \Phi_{,i}\, \Phi_{,j} \ , \end{aligned} \tag{2.4}$$

where i, j run over the spatial indices.

Usually we solve an equation such as (2.2) by means of a Green's function. Namely, we take the solution to be

$$h_G(x) := \int dx'\, G(x,x')\, Src(x') \ , \tag{2.5}$$

where in this case [2]

$$G(x,x') \propto \Phi(x-x') \ . \tag{2.6}$$

Divergences of the integral may come from regions where the integrand is infinite or from large regions of integration. In our case short-distance divergences are possible in the vicinity of $x' \to 0$ (there are none as $x' \to x$), and long-distance divergences are possible as $r \to \infty$. By a standard abuse of language we shall refer to the short-distance divergences as

[1] This form allows for non-linearities arising from both the Newtonian approximation $\sim \Phi$ and sources introduced through multipole matching $\sim \partial^l \Phi$. $k_2 \geq 2$ since the source is non-linear and $k_1 \geq 2$ from a dimensional analysis.

[2] We take the Green function with no boundary conditions at 0. One could have taken a more involved $G(x,x')$ with an arbitrary multipole distribution at $x=0$, but this would not cure the divergences which we will encounter. For the precise prefactor in (2.6) see (4.14).

"UV divergences" and to long-distance ones as IR even though the theory is not quantum and distances and energies are not reciprocal.

UV divergences. The integrand of (2.5) diverges as $\rho' \to 0$ due to the singularity in $Src(\rho')$ which is inherited from $\Phi(\rho')$. Assuming that for small ρ' the behavior of the singular and angle-independent ("S-wave") part of the integrand is $G(x,x')\,Src(x') \sim \rho'^{-s}$ we get from (2.5)

$$h_G(x) \simeq \int_0 d\rho'\,\rho'^{d-2}\,\rho'^{-s} \ . \tag{2.7}$$

Therefore we have UV divergences (for all x) exactly if $s \geq d-3$. For PN we see from (2.4) that $Src(\rho') \simeq \Phi_{,i}\,\Phi_{,j} \simeq \rho^{-2(d-2)}$, [3] and therefore *we indeed have UV divergences* for all d. At higher orders the divergences only get worse.

IR divergences. If for large r Src behaves as $Src(r) \sim r^{-s_\infty}$ we get

$$h_G(x) \simeq \int^\infty dr'\,r'^{d-3}\,\frac{1}{r'^{d-4}}\,r'^{-s_\infty} \ . \tag{2.8}$$

Therefore we would have divergences exactly if $s_\infty \leq 2$, but the slowest relevant decay rate of Src – a PN source in $d=5$ – is $s_\infty = 2(d-3) = 4$ and hence *there are no IR divergences.*

Unregulated multipoles. Even though the Green function integral diverges we can still find a particular solution of (2.2). Locally around the origin we may separate the equation by using radial variables (ρ,χ) and we find

$$\left(\partial_\rho{}^2 + \frac{d-2}{\rho}\,\partial_\rho - \frac{l(l+d-3)}{\rho^2}\right)\,h_l = Src_l \ . \tag{2.9}$$

The particular solution is

$$h_l \propto \rho^{-s_l+2} \ , \tag{2.10}$$

where the constants s_l are defined through $Src_l(\rho) \sim \rho^{-s_l}$, and one should allow for a possible addition of a homogeneous solution of the form $\rho^l,\ \rho^{-(d-3)-l}$.

Indeed, if particular solutions can be found in explicit form, and their homogeneous part determined by matching, then there is no problem of divergences, as was done in [7] for $d=5$. However, we are often interested only in the asymptotic form of the metric and then a Green-function-based multipole expansion is useful. For instance, in this paper we will compute only the corrections to the thermodynamic quantities, the mass and the tension, rather than the whole metric. Rephrasing from a different perspective, the particular solution (2.10) at small ρ does not give information about the large ρ asymptotics, and the separation of variables is not valid globally.

We now wish to re-formulate the divergence problem in terms of multipoles, which are particularly suited for divergences localized at a point. Since the divergences come from the vicinity of $x'=0$ we Taylor expand $G(x,x')$ there. The convergence radius ρ_{cnvg} would

[3] More precisely, one needs to find the degree of the singular and angle-independent part of $G(x,x')\,Src(x')$. Since for PN the leading singularity of $Src(\rho')$ has a quadrupole part then $s \geq 2(d-3)$ and the conclusion is unchanged.

be $\sim \min(L, x)$ and for $x \to \infty$ $\rho_{cnvg} \sim L$. Then we separate the domain of integration in (2.5) to within the convergence radius $\rho < \rho_{cnvg}$ and outside it. Outside there are no divergences, and inside we may perform a standard multipole expansion

$$h(x) \simeq \partial'_{i_1} \dots \partial'_{i_l} G(x,0) \int_{\rho<\rho_{cnvg}} dV'\, Src(x')\, x'^{i_1} \dots x'^{i_l} \ . \tag{2.11}$$

Thus we see that all the divergences may be encoded by the diverging multipoles at the origin, defined by the small ϵ behavior of

$$\mathcal{M}_l := \int_{|\rho|>\epsilon} dV\, Src(x)\, x^{i_1} \dots x^{i_l} \ . \tag{2.12}$$

Regularization. The matching boundary conditions (b.c.) suggest a natural regularization procedure: "cut-off and match". Namely, cut-off the integral (2.12) at $\rho = \epsilon$, adjust it by a constant (corresponding to adding an allowed homogeneous piece to h) to conform with the matching b.c., and finally take the limit $\epsilon \to 0$.

This idea is translated into formulae as follows. Expand the multipoles in a Laurent series in ϵ

$$\mathcal{M}_l(\epsilon) = \sum_{j=j_{min}}^{\infty} \mathcal{M}_{l,j}\, \epsilon^j \ , \tag{2.13}$$

where at order k dimensional analysis gives $j_{min} = -(k - l - (d-3))$. From the finite part multipoles, $\mathcal{M}_{l,0}$, we may construct a function h which is known in the mathematical literature as Hadamard's partie finie [14], or Hadamard's finite part, and hence we shall denote

$$\mathrm{Pf}(\mathcal{M}_l) := \mathcal{M}_{l,0} \tag{2.14}$$

(see also [15] for its use in the Post-Newtonian context). Alternatively, the regularization could be defined through analytic continuation and it is also closely related to Cauchy's Principal Value.

The divergent piece in (2.13), (namely the sum over $j < 0$), is interpreted as coming from the vicinity of the origin, and thus according to the matching b.c. should be replaced by an appropriate value of the multipole $\mathcal{M}_l^{match}$ read from the asymptotics ($\rho \gg \rho_0$) of the near zone metric. Thus we arrive at the expression for *the regularized multipoles* which is one of our central formulae

$$\boxed{\mathcal{M}_l \to {\mathcal{M}_l}^R = \mathrm{Pf}(\mathcal{M}_l) + \mathcal{M}_l^{match}} \ . \tag{2.15}$$

"Renormalization". This regularization procedure is reminiscent of renormalization in Quantum Field Theory. There we know that in order to renormalize a theory it must have as many free parameters, or counter-terms, to be set by experimental data, as divergences.[4]

[4]In QFT a theory is called normalizable if the number of counter-terms is finite, and non-renormalizable otherwise. In our case there will be infinitely many divergences, but all the necessary "experimental data" will come from matching.

Here by analogy we need

$$\#(\text{divergences}) = \#(\text{matchings}) \ . \tag{2.16}$$

However, at each order there are only certain (finitely many) matchings coming from the other zone, as encoded by the "perturbation ladder" (see figure 3). What would guarantee that (2.16) is satisfied? The answer turns out to be reminiscent of Quantum Field Theory as well – it is dimensional analysis! At order k in the asymptotic zone we are considering divergences of the form

$$\frac{{\rho_0}^k}{\rho^{l+d-3}\,\epsilon^a\,L^b} \ , \tag{2.17}$$

where $a > 0$, $b \geq 0$. Since all the quantities in this expression have dimensions of length, but the expression itself (namely h) is dimensionless then $a + b = k - l - (d-3)$ and in particular $k-(d-3)-l > 0$. However, this is exactly the condition for matching, as is seen by considering appropriate terms at order b in the near zone, namely taking $a = 0$, $b > 0$.

No self-interaction. We are also familiar with a different approach to regularization – the removal of self-interaction. The idea is to separate the black hole solution into a Schwarzschild piece and a correction coming from the compact dimensions. Intuitively one expects that all the divergences come from the singular Schwarzschild part, and that these will be absorbed automatically by adjusting its parameters.

This approach is commonly used in the Post-Newtonian context (see for example the review talk [16]). However, it is not known how to implement this idea at arbitrary orders of perturbation, but rather some results are known at low orders. We expect the "no-self-interaction" to be equivalent to our method and it would be interesting to demonstrate it. Our explicit calculations in the next chapters do lend support to this idea, as it turns out that if one omits singular products from the double sum over images in PN, it is equivalent to Hadamard's regularization. In other words, Hadamard's partie finie of the singular part is zero at this order.

Moreover, we observe another simplification whereby the matching is exactly cancelled by the multipole contribution of the $\mathcal{O}(\Phi^2)$ term within the Laplacian in (2.4). This may very well be a special feature of our gauge, and it would be interesting to determine whether it can be made to continue in higher orders as well. So altogether in our PN computation regularization may be replaced by discarding the singular self-interaction source term and matching is automatically cancelled.

The near zone. In the near zone the regularization of divergences works in parallel to the method just described for the asymptotic zone, so we only describe the main points. Actually, in this paper we shall not treat the near zone at all.

Here the linear operator, L, in (2.1) is given by the Heun equation (see [5]), which for $\rho \gg \rho_0$ reduces once more to the Laplacian. The divergences are IR rather than UV. Regularization proceeds by introducing a large-distance cut-off R and considering the Laurent series for multipoles as $R \to \infty$. After a partie finie regularization the matched value from the asymptotic zone should be added just as in (2.15).

3. Review of leading order results

3.1 The general procedure

Using the matching procedure described in [5] we obtained the leading corrections to the Schwarzschild metric of a black hole as a result of the compact dimension. The matching procedure for the leading corrections is summarized in figure 3.

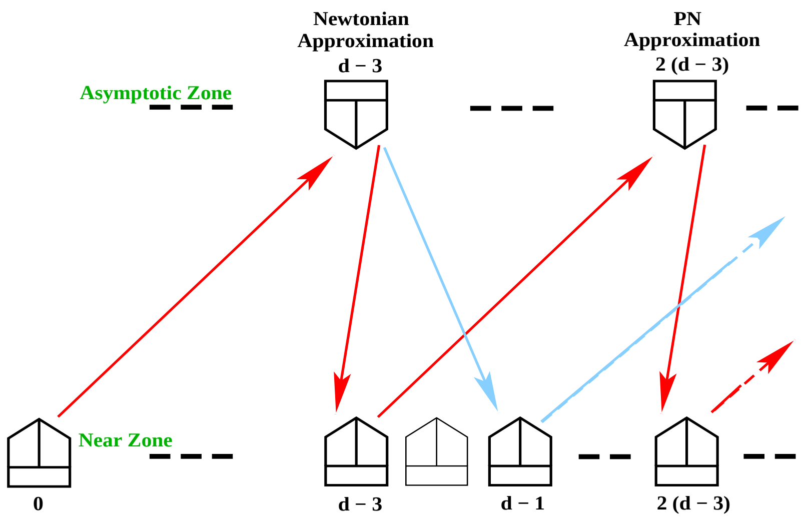

Figure 3: The first steps in the matching procedure. Each box in the top row denotes a specific order in the asymptotic zone, and the lower row depicts the near zone. Arrows denote the flow of matching information between the zones: Dark (red) arrows denote monopole matching ($l = 0$), light arrows (light-blue) denote the quadrupole ($l = 2$), and higher l arrows are not shown.

The metric in the asymptotic zone is expanded in the radius of the black hole ρ_0 in the following form

$$g_{\mu\nu}^{(\text{asymp})} = \eta_{\mu\nu} + h_{\mu\nu}^{[1]} + h_{\mu\nu}^{[2]} + ..., \tag{3.1}$$

where $h_{\mu\nu}^{[1]} \propto \rho_0^{d-3}$,$h_{\mu\nu}^{[2]} \propto \rho_0^{2\,(d-3)}$ and so on. Note that the matching procedure may introduce corrections which are not integer powers of ρ_0^{d-3} but this would happen only after the post-newtonian corrections.

According to the matching procedure, in order to determine the Post-Newtonian corrections (order $2\,(d-3)$ in figure 3) we first need to know the Newtonian approximation and the near zone leading order. Therefore we start here by reviewing the results that were obtained in [5] for these orders. The leading order in the asymptotic zone is simply the Newtonian potential while the leading order corrections in the near zone are decomposed into multipolar static perturbations to the Schwarzschild-Tangherlini metric (following Regge-Wheeler [17] and using a similar gauge).

For the post-Newtonian expansion we make a quite-standard gauge choice. One starts by writing the Ricci tensor as [18]

$$R_{\mu\nu} = -\frac{1}{2} g_{\mu\sigma} g_{\nu\rho} g^{\alpha\beta} \frac{\partial^2 g^{\rho\sigma}}{\partial x^\alpha \partial x^\beta} + \Gamma_\mu^{\alpha\beta} \Gamma_{\nu,\alpha\beta} - \Gamma_{\mu\nu}, \tag{3.2}$$

where $\Gamma_{\mu,\alpha\beta}$ and $\Gamma_{\mu}^{\alpha\beta}$ are the Christoffel symbols of the first and the second kind, respectively, and in addition one defines

$$\Gamma^{\nu} := g^{\alpha\beta}\Gamma^{\nu}_{\alpha\beta}\,,$$
$$\Gamma_{\mu\nu} := \frac{1}{2}\left(g_{\mu\rho}\frac{\partial\Gamma^{\rho}}{\partial x^{\nu}} + g_{\nu\sigma}\frac{\partial\Gamma^{\sigma}}{\partial x^{\mu}} - g_{\mu\rho}g_{\nu\sigma}\frac{\partial g^{\rho\sigma}}{\partial x^{\alpha}}\Gamma^{\alpha}\right).$$

Next one chooses the Harmonic (or de Donder [5]) gauge by the requirement that

$$\Gamma^{\nu} = \Box x^{\nu} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\beta}}(\sqrt{-g}g^{\beta\nu}) \equiv 0\,, \tag{3.3}$$

where we denote by g the determinant of the metric $g_{\mu\nu}$. In this gauge, the last term in the expression of the Ricci tensor above vanishes.

3.2 Newtonian approximation

The first step in the iterative procedure was to solve the equations in the Newtonian approximation (see for example [5, 6, 20])

$$h^{[1]}_{tt} = \Phi, \tag{3.4}$$
$$h^{[1]}_{ij} = \frac{1}{d-3}\,\Phi\,\delta_{ij}, \tag{3.5}$$

and

$$\Phi := \rho_0^{d-3}\sum_{n=-\infty}^{\infty}\frac{1}{(r^2+(z+nL)^2)^{\frac{d-3}{2}}}, \tag{3.6}$$

where the Latin indices stand for the spatial components. In order to match this approximation with the near zone we have to expand Φ around $\rho = 0$ in polar coordinates (ρ,χ)

$$\Phi(r,z) = \frac{\rho_0^{d-3}}{\rho^{d-3}} + 2\cdot\frac{\rho_0^{d-3}}{L^{d-3}}\,\zeta(d-3) + \frac{(d-3)\,(d-2)\,\rho^2\,\rho_0^{d-3}}{L^{d-1}}\,\zeta(d-1)\,\Pi_0^{2,d}(\chi) + \mathcal{O}(\rho^4), \tag{3.7}$$

where ζ is Riemann's zeta function[6] and

$$\Pi_0^{2,d}(\chi) := \frac{1}{d-2}\,\left(\cos^2(\chi)\,(d-1)-1\right),$$

which is the generalized Legendre function that corresponds to the quadrupole. In particular, the overall prefactor in (3.6) is set by the matching described by the left most arrow in figure 3.

From the asymptotic form of the metric (monopole terms), when $r \gg z$,

$$g_{\mu\nu} = \frac{c_{\mu\nu}}{r^{d-4}} + \mathcal{O}\left(\frac{1}{r^{d-3}}\right), \tag{3.8}$$

[5] The first introduction of this gauge appeared in [19].

[6] Riemann's zeta function is defined as $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$.

one can extract the mass and tension [10, 21] defined through the standard first law

$$dM = \frac{\kappa}{8\pi}\, dA + \tau dL \ . \tag{3.9}$$

(For weak sources this definition is equivalent to $M := \int dV_{d-1} T_{tt}$, $\tau := -\int dV_{d-1} T_{zz}/L$.)

In harmonic gauge the asymptotic measurable quantities can be expressed in terms of the constants $c_{\mu\nu}$ in the asymptotics as follows [10, 21]

$$M = \frac{\Omega_{d-3}}{16\,\pi\, G_d}\left((d-3)\, c_{tt} - c_{zz}\right), \tag{3.10}$$

$$\tau = \frac{\Omega_{d-3}}{16\,\pi\, G_d\, L}\left(c_{tt} - (d-3) c_{zz}\right), \tag{3.11}$$

where

$$\Omega_{d-3} = \frac{2\,\pi^{\frac{d-2}{2}}}{\Gamma(\frac{d-2}{2})},$$

is the area of a unit S^{d-3}.

At this order, the asymptotic constants can be determined from the asymptotic expansion[7] ($r \gg z$) of the Newtonian potential

Φ

$$\Phi = \frac{\Omega_{d-2}}{\Omega_{d-3}} \cdot \frac{d-3}{d-4}\, \rho_0^{d-3}\, \frac{1}{r^{d-4}} + \mathcal{O}\left(\frac{1}{r^{d-3}}\right). \tag{3.12}$$

Then

$$c_{tt}^{[1]} = \frac{\Omega_{d-2}}{\Omega_{d-3}} \cdot \frac{d-3}{d-4}\, \rho_0^{d-3}, \tag{3.13}$$

$$c_{zz}^{[1]} = \frac{\Omega_{d-2}}{\Omega_{d-3}} \cdot \frac{1}{d-4}\, \rho_0^{d-3}. \tag{3.14}$$

Thus, the tension vanishes in the first order approximation and for the mass we obtain

$$M = \frac{(d-2)\Omega_{d-2}}{16\pi\, G_d}\, \rho_0^{d-3} + \mathcal{O}\left(\rho_0^{2(d-3)}\right). \tag{3.15}$$

3.3 The near zone metric

In the near zone we obtained the first monopole correction to the Schwarzschild metric by matching with the Newtonian approximation [5]. The metric in the near zone reads

$$\begin{aligned}
g_{tt,d}^{\text{near}} &= - \left(1 - \frac{\rho_0^{d-3}}{\rho_s^{d-3}}\right)\left(1 - \frac{2\,\zeta(d-3)\,\rho_0^{d-3}}{L^{d-3}}\right) + \mathcal{O}\left(\frac{1}{L^{d-2}}\right), \\
g_{\rho\rho,d}^{\text{near}} &= \left(1 - \frac{\rho_0^{d-3}}{\rho_s^{d-3}}\right)^{-1}\left(1 + \frac{2\,\zeta(d-3)\,\rho_0^{d-3}}{(d-3)\,L^{d-3}}\right) + \mathcal{O}\left(\frac{1}{L^{d-2}}\right), \\
g_{\chi\chi,d}^{\text{near}} &= \rho_s^2\left(1 + \frac{2\,\zeta(d-3)\,\rho_0^{d-3}}{(d-3)\,L^{d-3}}\right) + \mathcal{O}\left(\frac{1}{L^{d-2}}\right).
\end{aligned} \tag{3.16}$$

The metric in the near zone is written in Schwarzschild coordinates when the metric in the asymptotic zone is written in Harmonic coordinates (ones which satisfy (3.3)). We

[7]Which may be gotten from flux conservation for $\vec{\nabla}\Phi$.

denote by ρ_s the radial coordinate in Schwarzschild coordinates and by ρ the radial coordinate in the Harmonic coordinates. In order to match them we have to use the leading terms of the transformation rule [5]

$$\rho = \rho_s - \frac{\rho_0^{d-3}}{2\,(d-3)\,\rho_s^{d-4}} + \mathcal{O}(\frac{1}{\rho_s^{d-3}}). \tag{3.17}$$

and get the near zone metric in Harmonic coordinates

$$\begin{aligned} g_{tt}^{\text{near}} &= -\left(1-\frac{\rho_0^{d-3}}{\rho^{d-3}}\right)\left(1-\frac{2\,\zeta(d-3)\,\rho_0^{d-3}}{L^{d-3}}\right)+\mathcal{O}\left(\frac{1}{\rho^{2(d-3)}},\frac{1}{L^{d-2}}\right), \\ g_{rr}^{\text{near}} &= g_{zz}^{\text{near}} = \left(1+\frac{\rho_0^{d-3}}{(d-3)\,\rho^{d-3}}\right)\left(1+\frac{2\,\zeta(d-3)\,\rho_0^{d-3}}{(d-3)\,L^{d-3}}\right)+\mathcal{O}\left(\frac{1}{\rho^{2(d-3)}},\frac{1}{L^{d-2}}\right). \end{aligned} \tag{3.18}$$

Note that the prefactor of $\frac{1}{\rho^{d-3}}$ in the near zone determines the monopole corrections to the asymptotic zone metric through the matching procedure. In particular, the term that behaves as $\frac{\rho_0^{2(d-3)}}{\rho^{d-3}\,L^{d-3}}$ is the one that should be matched with the post-Newtonian corrections to the metric in the asymptotic zone (see figure 3).

Higher multipoles.

From the "perturbation ladder" (figure 3) we see that proceeding in the near zone beyond the leading correction (order $d-3$) we have contributions of higher multipoles that are determined from the Newtonian potential, before the first non-linear iteration at order $2(d-3)$. The multipole moments of the newtonian potential are matched with the multipole linear perturbations of the BH. In our case, only even multipole numbers contribute corrections to the metric due to the $\chi \to \pi - \chi$ symmetry. Any even multipole number $l < d-3$ contributes a correction to order $l+d-3$ in L^{-1} before order $2\,(d-3)$ (see [5]), where the next monopole correction enters. For example, in $d=5$ there are no corrections of higher multipoles before order 4, in $d=6$ we have a quadrupole correction $l=2$ and so on.

For any $l < d-3$ we obtained in [5] explicit expressions for the correction to the near zone metric. We review here the final results. The correction of order $l+d-3$ can be expressed in the form

$$g_{\mu\nu}^{(\text{near},l+d-3)}\,dx^\mu\,dx^\nu = g_{tt}^{(\text{near},l+d-3)}\,dt^2 + g_{\rho_s\rho_s}^{(\text{near},l+d-3)}\,d\rho_s^2 + g_{\chi\chi}^{(\text{near},l+d-3)}\,d\Omega_{d-2}^2, \tag{3.19}$$

where

$$g_{\chi\chi}^{(\text{near},l+d-3)} = \rho_s^2\,c_l\,\rho_0^l\,\frac{\Gamma(1+\frac{l}{d-3})\,\Gamma(2+\frac{l}{d-3})}{\Gamma(1+\frac{2\,l}{d-3})\,(l-1)}\,E_l(X)\,\Pi_0^{l,d}(\chi), \tag{3.20}$$

$$\begin{aligned} E_l(X) &= \left(1-(d-3)\,X\,\frac{d}{d\,X}\right)\left((1-X)\,{}_2F_1(1-\frac{l}{d-3},2+\frac{l}{d-3},2\,;1-X)\right), \\ X &:= \frac{\rho_s^{d-3}}{\rho_0^{d-3}}, \end{aligned} \tag{3.21}$$

${}_2F_1$ is the hypergeometric function and $\Pi_0^{l,d}(\chi)$ are the generalized Legendre polynomials given by a Rodriguez formula

$$\Pi_0^{l,d}(\chi) = \frac{\Gamma(\frac{d}{2}-1)}{\Gamma(l+\frac{d}{2}-1)}\,\sin^{4-d}(\chi)\,\left(\frac{1}{\sin(\chi)}\,\frac{d}{d\chi}\right)^l\,\sin^{2l+d-4}(\chi). \tag{3.22}$$

The constants c_l come from the metric in the asymptotic zone, namely, from the Newtonian potential. These are the constants that appear in

$$\frac{g_{\chi\chi}^{(\mathrm{asymp},d-3)}}{\rho^2} = \frac{\Phi}{d-3} = \frac{1}{d-3}\,\frac{\rho_0^{d-3}}{\rho^{d-3}} + \sum_{l=0}^{\infty} c_l\,\rho^l\,\Pi_0^{l,d}(\chi)\ , \tag{3.23}$$

- see (3.7) for the explicit values of c_l for $l=0,2$. The other two components of the metric correction are determined from $g_{\chi\chi}^{(\mathrm{near},l+d-3)}$

$$\begin{aligned} g_{tt}^{(\mathrm{near},l+d-3)} &= c_l\,\rho_0^l\,\frac{\Gamma(1+\frac{l}{d-3})\,\Gamma(2+\frac{l}{d-3})}{\Gamma(1+\frac{2\,l}{d-3})\,(l-1)}\,f\,A_l(\rho_s)\,\Pi_0^{l,d}(\chi), \\ & \qquad\qquad\qquad\qquad\qquad\qquad l\,<\,d-3 \\ g_{\rho\rho}^{(\mathrm{near},l+d-3)} &= c_l\,\rho_0^l\,\frac{\Gamma(1+\frac{l}{d-3})\,\Gamma(2+\frac{l}{d-3})}{\Gamma(1+\frac{2\,l}{d-3})\,(l-1)}\,\frac{B_l(\rho_s)}{f}\,\Pi_0^{l,d}(\chi), \end{aligned} \tag{3.24}$$

where

$$\begin{aligned} f &:= 1-\frac{\rho_0^{d-3}}{\rho_s^{d-3}}, \\ A_l(\rho_s) &= \frac{(d-2)\,\rho_s\,f}{2\,l\,(l+d-3)}\left[(d-2)\,\frac{d\,E_l}{d\rho_s} + \rho_s\,\frac{d^2\,E_l}{d\rho_s^2}\right] + \frac{d-4}{2}E_l, \end{aligned} \tag{3.25}$$

and B_l is obtained from the algebraic relation

$$B_l = -A_l - (d-4)\,E_l.$$

4. Post-Newtonian Thermodynamics

4.1 Post-Newtonian equations

Using the results for $h_{\mu\nu}^{[1]}$ (the Newtonian approximation) we can write Einstein's equations for $h_{\mu\nu}^{[2]}$, i.e. the Post-Newtonian equations, in the following form

$$\begin{aligned} \triangle\left(h_{tt}^{[2]} + \frac{1}{2}\Phi^2\right) &= 0, \\ \triangle\left(h_{ij}^{[2]} - \frac{1}{2}\delta_{ij}\left(\frac{\Phi}{d-3}\right)^2\right) &= -\frac{d-2}{2(d-3)}\Phi_{,i}\Phi_{,j}\quad . \end{aligned} \tag{4.1}$$

The origin contains a singular source which will be accounted for by matching rather than by introducing a point-like source.

Let us look at the "source" term (the RHS) for the spatial components. This term is proportional to

$$\Phi_{,i}\Phi_{,j} = (d-3)^2 \rho_0^{2\,(d-3)} \sum_{m,n\in\mathbb{Z}} \frac{(x_i + m\,L\,\delta_{iz})\ (x_j + n\,L\,\delta_{jz})}{(r^2+(z+m\,L)^2)^{\frac{d-1}{2}}\ (r^2+(z+n\,L)^2)^{\frac{d-1}{2}}}.$$

In this sum we see the effect of interaction between the black hole and its mirror images, including the interaction between the mirror images themselves. One term in this sum is different - the term that corresponds to the self-interaction of the black hole, namely, when $m = n = 0$. This term exists in the post-Newtonian equations regardless of the boundary conditions. It has a singular behavior near the origin being proportional to $\frac{x_i x_j}{\rho^{2\,(d-1)}}$. For $i = j$ its integral over the volume diverges like $\frac{1}{\varepsilon^{d-3}}$ for $\varepsilon \to 0$.

We shall follow the "no self-interaction" approach together with "matching regularization" (see the discussion in section 2). Since the equations are linear we can separate the equations to the self-interaction part (SI) and the regular part (REG) which consists of the interaction with the mirror images. In order to keep the periodic boundary conditions in the compact dimension, we include in the SI part the self-interaction terms of the mirror images even though they are not singular in the domain $|z| \le L/2$.

Accordingly we separate

$$h^{[2]}_{\mu\nu} = h^{[2]\,SI}_{\mu\nu} + h^{[2]\,REG}_{\mu\nu}. \tag{4.2}$$

The separated PN equations are

$$\triangle\left(h^{[2]\,SI}_{tt} + \frac{1}{2}\Phi^2_{SI}\right) = 0, \tag{4.3}$$

$$\triangle\left(h^{[2]\,SI}_{ij} - \frac{1}{2\,(d-3)^2}\delta_{ij}\Phi^2_{SI}\right) = -\frac{d-2}{2(d-3)}\left(\Phi_{,i}\Phi_{,j}\right)_{SI}, \tag{4.4}$$

where

$$\Phi^2_{SI} = \rho_0^{2(d-3)} \sum_{n\in\mathbb{Z}} \frac{1}{(r^2+(z+n\,L)^2)^{d-3}},$$

$$\left(\Phi_{,i}\Phi_{,j}\right)_{SI} = (d-3)^2 \rho_0^{2\,(d-3)} \sum_{n\in\mathbb{Z}} \frac{(x_i + n\,L\,\delta_{iz})\ (x_j + n\,L\,\delta_{jz})}{(r^2+(z+n\,L)^2)^{d-1}},$$

and

$$\triangle\left(h^{[2]\,REG}_{tt} + \frac{1}{2}\Phi^2_{REG}\right) = 0, \tag{4.5}$$

$$\triangle\left(h^{[2]\,REG}_{ij} - \frac{1}{2\,(d-3)^2}\delta_{ij}\Phi^2_{REG}\right) = -\frac{d-2}{2(d-3)}\left(\Phi_{,i}\Phi_{,j}\right)_{REG}, \tag{4.6}$$

where

$$\Phi^2_{REG} = \rho_0^{2(d-3)} \sum_{m\neq n} \frac{1}{(r^2+(z+m\,L)^2)^{\frac{d-3}{2}}\ (r^2+(z+n\,L)^2)^{\frac{d-3}{2}}},$$

$$(\Phi_{,i}\Phi_{,j})_{REG} = (d-3)^2\rho_0^{2\,(d-3)} \sum_{m\neq n} \frac{(x_i+m\,L\,\delta_{iz})\ (x_j+n\,L\,\delta_{jz})}{(r^2+(z+m\,L)^2)^{\frac{d-1}{2}}\ (r^2+(z+n\,L)^2)^{\frac{d-1}{2}}},$$

$$(\,m,n\in\mathbb{Z}\,)\,.$$

Finally, the boundary conditions determining the homogeneous solution are supplied through matching with the near zone metric (3.18), as usual.

The self interaction part – SI. An explicit form of a particular solution for the singular part (4.3,4.4) can be found. For this purpose let us look at the equations in the case of a single source without any mirror images ($n=0$), whose metric we denote by $h^{[2]\,0}_{\mu\nu}$

$$\triangle\left(h^{[2]\,0}_{tt}+\frac{\rho_0^{2(d-3)}}{2\,\rho^{2(d-3)}}\right) = 0, \tag{4.7}$$

$$\triangle\left(h^{[2]\,0}_{ij}-\frac{1}{2\,(d-3)^2}\frac{\rho_0^{2(d-3)}}{\rho^{2(d-3)}}\delta_{ij}\right) = -\frac{(d-2)\,(d-3)}{2}\,\rho_0^{2(d-3)}\,\frac{x_i\,x_j}{\rho^{2\,(d-1)}}. \tag{4.8}$$

The solution for the equations above is

$$h^{[2]\,0}_{tt} = -\frac{\rho_0^{2(d-3)}}{2\,\rho^{2(d-3)}}, \tag{4.9}$$

$$h^{[2]\,0}_{ij} = \rho_0^{2\,(d-3)}\left[\frac{\delta_{ij}}{2\,(d-3)^2\,\rho^{2\,(d-3)}}+\begin{cases}\frac{\ln(\rho)}{8\,\rho^4}\left(\frac{4\,x_i\,x_j}{\rho^2}-\delta_{ij}\right)+\frac{x_i\,x_j}{12\,\rho^6}-\frac{11\,\delta_{ij}}{96\,\rho^4} & d=5\\ \frac{1}{4\,\rho^{2\,(d-3)}\,(d-5)}\left(\frac{\delta_{ij}}{d-3}-\frac{x_i\,x_j\,(d-3)}{\rho^2}\right) & d>5\end{cases}\right].$$

Now, the solution to the full SI equations (4.3,4.4) can be obtained by taking $z\to z+n\,L$ in (4.9) and summing over $n\in\mathbb{Z}$

$$h^{[2]\,SI}_{tt} = \sum_{n=-\infty}^{\infty} h^{[2]\,0}_{tt}(r,z+n\,L), \tag{4.10}$$

$$h^{[2]\,SI}_{ij} = \sum_{n=-\infty}^{\infty} h^{[2]\,0}_{ij}(r,z+n\,L). \tag{4.11}$$

The regular part – REG. The particular solution in the asymptotic zone can be expressed by an integral over Green's function

$$h^{[2]\,REG}_{tt} = -\frac{1}{2}\Phi^2_{REG}, \tag{4.12}$$

$$h^{[2]\,REG}_{ij} = \frac{1}{2\,(d-3)^2}\,\delta_{ij}\,\Phi^2_{REG}-\frac{d-2}{2\,(d-3)}\int(\Phi_{,i}\Phi_{,j})_{REG}\,(r',z')G(r-r',z-z')d\,V_{d-1}, \tag{4.13}$$

$$
\begin{array}{r|cc}
 & h_{tt} & h_{zz} \\
\hline
SI & 0 & 0 \\
REG\ \Phi^2 \text{ shift} & -\frac{1}{2}\, c_\Phi & +\frac{1}{2\,(d-3)^2}\, c_\Phi \\
REG \text{ Green's} & 0 & 0 \\
\hline\hline
\text{order [1] near} & -2\,\zeta & +\frac{2\,\zeta}{(d-3)^2}
\end{array}
\tag{4.16}
$$

Table 1: Matching budget for both h_{tt} and h_{zz} at Post-Newtonian order, namely for the coefficients of $\mathcal{O}\left(\rho_0^{2(d-3)}/(\rho^{d-3}\,L^{d-3})\right)$. At PN these quantities have 3 contributions SI – Self-interaction solutions see (4.9); REG Φ^2 shift – namely the contribution from Φ^2_{REG} to the regular term see (4.12,4.13,4.17). Here $c_\Phi := 4\zeta$ is the coefficient of the relevant term in Φ^2_{REG} (4.17) and in this table $\zeta := \zeta(d-3)$; REG Green's - for h_{zz} the Green's function integral in (4.13) has no relevant term according to (4.18). For h_{tt} this term is identically zero (4.12). In the fourth line we see the leading order results for the near zone, taken from (3.17), and we confirm that they are identical to the sum of the 3 items above them and therefore the matching constants vanish.

where Green's function is

$$
G(r-r', z-z') = -\frac{1}{(d-3)\Omega_{d-2}} \sum_{n=-\infty}^{\infty} \frac{1}{\left((r-r')^2 + (z-z'+n\,L)^2\right)^{\frac{d-3}{2}}}. \tag{4.14}
$$

4.2 Matching

We still need to match the solutions with the near zone, a matching which will turn out to vanish. As encoded in the "ladder" figure (figure 3) we should match the monopole term at order $2(d-3)$ in the asymptotic zone, with the monopole of order $d-3$ in the near zone. Accordingly, the relevant term in the double expansion in the overlap region $\rho_0 \ll \rho \ll L$ is

$$
\frac{\rho_0^{2(d-3)}}{\rho^{d-3}\,L^{d-3}}, \tag{4.15}
$$

The matching "budget" is composed of several items summarized in the table 1, which we proceed to explain.

- SI — In the case of the Self-interaction equations we have an explicit solution (4.9). One sees that neither $h_{tt}^{[2]}$ nor $h_{zz}^{[2]}$ contain any term of the form (4.15).

  In the regular solutions (4.12,4.13), we distinguish two parts, the first term is just a shift proportional to Φ^2_{REG}, which we refer to as "REG Φ^2 shift", and the second is an integral with a Green function kernel (there is no such term in (4.12)), which we refer to as "REG Green's".

- REG Φ^2 shift — In Φ^2_{REG} we have a term of the relevant form (4.15)

$$
\Phi^2_{REG} = \frac{4\,\rho_0^{2\,(d-3)}}{\rho^{d-3}\,L^{d-3}}\,\zeta(d-3) + \frac{4\,\rho_0^{2\,(d-3)}}{L^{2\,(d-3)}} \left(\zeta^2(d-3) - \frac{1}{2}\,\zeta\,(2\cdot(d-3))\right) + ..., \tag{4.17}
$$

whose coefficient we denote by $c_\Phi \equiv 4\,\zeta(d-3)$, where we used the expansion for Φ (3.7). We write here explicitly only the two monopole terms, i.e. terms which are independent of the angle χ.

- REG Green's — Next we show that the piece of $h^{[2]}_{zz}$ in (4.13) involving an integral with Green's function

$$\int (\Phi_{,i}\Phi_{,j})_{REG}\,(r',z')G(r-r',z-z')d\,V_{d-1}.$$

does not contain terms of the form (4.15). For this purpose let us look at the leading term near $\rho=0$ in the integral

$$\int_0^\infty (\Phi_{,i}\Phi_{,j})_{REG}\,(\rho')G(\rho,\rho')\rho'^{d-2}\,d\,\rho' \simeq \int_\rho^\infty \frac{1}{\rho'^{d-3}}\,\frac{1}{\rho'^{d-3}}\,\rho'^{d-2}\,d\,\rho' \simeq \frac{1}{\rho^{d-5}}\ , \tag{4.18}$$

where in the first equality we estimated the singular angle-independent part of $(\Phi_{,i}\Phi_{,j})_{REG}$ by $1/\rho'^{d-3}$ and replaced $1/|x-x'|^{d-3}$ in the Green function by $1/\rho'^{d-3}$ with a cutoff at ρ. Thus, we find that the leading behavior when $\rho\to 0$ is $\mathcal{O}\left(\rho^{-(d-5)}\right)$, namely there are no terms that behave as (4.15).

Adding up these 3 contributions for both $h^{[2]}_{tt}$ and $h^{[2]}_{zz}$ we find exact agreement with the near zone expressions (3.18), as seen in table 1. Therefore the matching constants indeed vanish.

4.3 Results

We did not find an analytic solution for the integral (4.13), and hence we do not have a full analytic form for the metric at this order. However, we can analytically evaluate the asymptotic form of this metric, thus determining the PN corrections to the thermodynamic quantities, M, τ. The results agree with Harmark's results [6] which were obtained using a different, single patch, method.

Our goal is to find $c^{[2]}_{tt}$ and $c^{[2]}_{zz}$, the next to leading corrections to g_{tt} and g_{zz} for $r\gg z$. From the asymptotic form (3.12) we see that the terms in the expansion of Φ^2 for $r\gg z$ do not contribute to the asymptotic constants. Moreover, we found in the last subsection that the matching constants vanish. Thus only the source terms (the RHS in (4.1)) contribute and we have

$$c^{[2]}_{tt} = 0, \tag{4.19}$$

$$c^{[2]}_{zz} = \frac{d-2}{2\,(d-3)\,(d-4)\,\Omega_{d-3}}\,\mathrm{Pf}\left(\int (\Phi_{,z})^2\,d\,V_{d-1}\right). \tag{4.20}$$

Pf stands for the finite part of the divergent integral according to Hadamard's partie finie regularization.

On dimensional grounds

$$\mathrm{Pf}\left(\int (\Phi_{,z})^2\,d\,V_{d-1}\right) = const\,\frac{\rho_0^{\,2(d-3)}}{L^{d-3}}\ , \tag{4.21}$$

where $const = const(d)$. We evaluated these constants by separating the integral into SI and REG pieces. The partie finie of the SI part vanishes and we find (the detailed calculation of the integral and its regularization appears in Appendix A)

$$\mathrm{Pf}\left(\int (\Phi_{,z})^2\, dV_{d-1}\right) = -\frac{\rho_0^{2\,(d-3)}}{L^{d-3}}\Omega_{d-2}(d-4)(d-3)\zeta(d-3)\ . \tag{4.22}$$

Substituting back into (4.20) gives us the final expressions

$$c_{tt}^{[2]} = 0, \tag{4.23}$$

$$c_{zz}^{[2]} = -\frac{\rho_0^{2\,(d-3)}}{L^{d-3}}\,\frac{\zeta(d-3)\,(d-2)\,\Omega_{d-2}}{2\,\Omega_{d-3}}. \tag{4.24}$$

Using (3.10,3.11) and adding it to the results in the Newtonian approximation (3.15) we obtain *the "measurable" quantities* as

$$\boxed{\begin{aligned} M &= \tfrac{(d-2)\Omega_{d-2}}{16\pi\, G_d}\,\rho_0^{d-3}\left(1+\tfrac{\zeta(d-3)}{2}\,\tfrac{\rho_0^{d-3}}{L^{d-3}}\right)+\mathcal{O}\left(\rho_0^{3(d-3)}\right) \quad , \\ \tau\, L &= \quad \tfrac{(d-3)\,(d-2)\,\Omega_{d-2}\,\zeta(d-3)}{32\pi\, G_d}\,\tfrac{\rho_0^{2\,(d-3)}}{L^{d-3}}+\mathcal{O}\left(\rho_0^{3(d-3)}\right) \quad . \end{aligned}} \tag{4.25}$$

4.4 Newtonian derivation of tension to leading order

We shall now show that the leading order tension (4.25) can be understood from Newtonian gravity. From the first law of black hole thermodynamics (3.9) we can express the tension as

$$\tau = \left(\frac{d\,M}{d\,L}\right)_S \ , \tag{4.26}$$

namely, the change of the mass in reaction to a change in the period of the compact dimension when we keep the entropy constant. The demand for constant entropy of a small black hole just means that ρ_0 (as well as the rest-mass M_0) are fixed. The black hole mass in the Newtonian limit is

$$M = M_0 + U\ , \tag{4.27}$$

where the mass is expressed as a sum of M_0, the intrinsic mass of the black hole given by the leading order of (4.25)

$$M_0 = \frac{(d-2)\Omega_{d-2}}{16\pi\, G_d}\,\rho_0^{d-3}\ , \tag{4.28}$$

$U \ll M_0$ is the gravitational potential energy, written as a sum over all mirror images with the standard prefactor 1/2 to avoid over-counting

$$U = \frac{1}{2}\sum_{n\,\in\,\mathbb{Z}\backslash\{0\}} U_n = \sum_{n\,\in\,\mathbb{N}} U_n\ . \tag{4.29}$$

The (negative) Newtonian gravitational potential energy is given by

$$U = -\frac{1}{2}\,h_{tt}\,M_0\ . \tag{4.30}$$

We may motivate this expression from two perspectives: geodesic motion or the red-shift of energy.

From the geodesic motion point of view U is obtained by integration over the Newtonian force that is needed to bring the black hole from infinity. The Newtonian force is determined from the Geodesic equation

$$\ddot{x}^\alpha + \Gamma^\alpha_{\mu\nu}\,\dot{x}^\mu \dot{x}^\nu = 0, \tag{4.31}$$

that in the Newtonian limit (weak field limit) gives

$$\frac{d^2\,x^i}{d\,t^2} = \frac{1}{2}\,h^{[1]}_{tt,i}\ , \tag{4.32}$$

where $h^{[1]}_{tt}$ is given in (3.4). Comparing with Newton's second law we write the Newtonian gravitational field as

$$\vec{E} = \frac{1}{2}\,\nabla h_{tt}, \tag{4.33}$$

thereby motivating (4.30).

Alternatively, the energy red-shift is given by

$$\begin{aligned} U &= M - M_0 \approx \partial_t - \partial_\tau \approx \left(1 - \sqrt{-g^{tt}}\right)\,\partial_t \\ &\approx \left(1 - \sqrt{-g^{tt}}\right)\,M_0 = \left(1 - \sqrt{1 + h_{tt}}\right)\,M_0 \simeq -\frac{1}{2}\,h_{tt}\,M_0\ . \end{aligned} \tag{4.34}$$

Now we may substitute

$$U_n = -\frac{1}{2}\,\frac{\rho_0^{d-3}\,M_0}{(n\,L)^{d-3}},$$

and find

$$\tau\,L = L\,\frac{d\,U}{d\,L} = -L\,\frac{d}{dL}\sum_{n\,\in\,\mathbb{N}}\frac{\rho_0^{d-3}\,M_0}{2\,n^{d-3}\,L^{d-3}} = \frac{(d-3)\,\zeta(d-3)}{2}\,\frac{\rho_0^{d-3}}{L^{d-3}}\,M_0. \tag{4.35}$$

Finally, substituting the expression for the rest mass (4.28) we recover the tension formula (4.25).

5. Implications for the phase diagram

In this section we shall translate the PN thermodynamic constants which we obtained into the phase diagram. The tension formula will dictate the slope of the small black hole branch. By extrapolation we shall attempt to learn about the phase transition region. Following the prediction for a critical dimension in this system [2], we shall explore the dimensional dependence of the extrapolation and indeed we shall find an indication for Sorkin's critical dimension [13].[8]

[8]This extrapolation could have been done ever since the results were first found in [6].

For the phase diagram it is convenient to define a dimensionless mass to serve as a control parameter

$$\mu := \frac{G_d\, M}{L^{d-3}}.$$

From the expression for the tension (4.25) we obtain an estimated $\widetilde{\mu}$ as a function of the dimensionless tension n for small black holes

$$\widetilde{\mu}_{BH} = \frac{(d-2)\Omega_{d-2}}{8\,\pi\,(d-3)\,\zeta(d-3)}\, n, \tag{5.1}$$

where the dimensionless tension n is defined by

$$n := \frac{\tau\, L}{M}.$$

This quantity serves us as a good dimensionless order parameter for the phase diagram [10, 20], and on the uniform black-string branch it attains the constant value

$$n_{st} = \frac{1}{d-3}. \tag{5.2}$$

In figure 4 we give the mass μ against the tension n for $d = 6$ for both the analytic linear approximation (5.1) and the numerical results for the whole phase diagram (the black hole and the non-uniform black string) taken from [12] (where the results of [11, 22] are incorporated[9]) . For the small black hole branch (lower right) we see excellent agreement.

Next we would like to extrapolate and define the extrapolated intersection point of the linear approximation with the black hole branch

$$\mu_X := \widetilde{\mu}_{BH}(n_{st}) = \frac{(d-2)\Omega_{d-2}}{8\,\pi\,(d-3)^2\,\zeta(d-3)}\ , \tag{5.3}$$

(see also figure 4). Now we can compare μ_X with the Gregory-Laflamme (GL) critical mass μ_{crit} as a function of the dimension. In figure 5 we plot μ_X/μ_{crit} for various d. The critical mass was computed from k_{GL}, the critical wave-number through

$$\mu_{crit} = \frac{(d-3)\,\Omega_{d-3}}{16\,\pi}\,\left(\frac{k_{GL}}{2\,\pi}\right)^{d-4}\ , \tag{5.4}$$

and values of k_{GL} for various dimensions were obtained in [13] and can be found in [23].

For large d, k_{GL} behaves as $k_{GL} \sim \sqrt{d}$ and the ratio μ_X/μ_{crit} tends to zero strongly. A similar observation appears in [4]. We interpret that as an indication for a second order phase transition, where the black hole phase does not reach the Gregory-Laflamme region, but rather turns first into a stable non-uniform string phase, and only the latter joins into the GL point. Since [24] it is known that at $d = 5$ the transition is first order. Thus we find an indication for a critical dimension, which separates first and second order behavior and was indeed discovered by Sorkin to lie at $D^* =$ "13.5" [13]. Interestingly, our graph shows a maximum in the range $12 \leq d \leq 14$ which is the location of the critical dimension D^*.

[9]We are grateful to T. Wiseman and H. Kudoh for sending us their data.

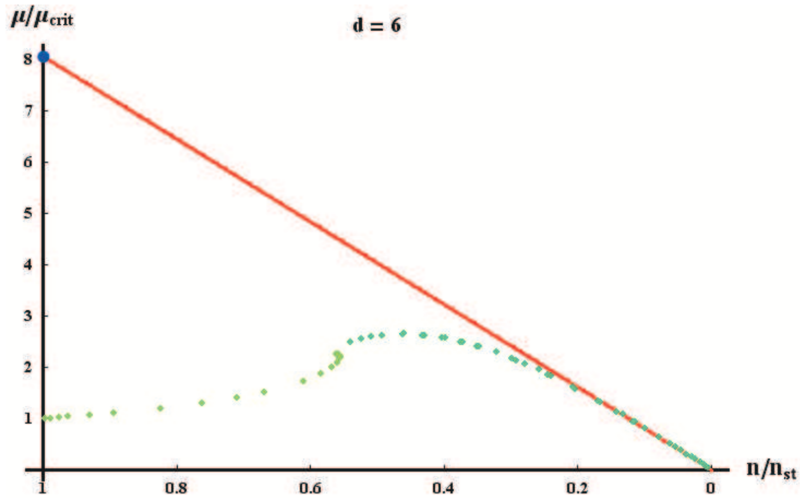

Figure 4: The analytic linear approximation line (red) compared to the numerical data points (green) of [12] for the complete phase diagram in $d = 6$ consisting of two branches: the black hole and the non-uniform black string. The vertical axis is proportional to the mass, and the horizontal axis to the tension (see text for exact definitions). For small black holes (lower right) we see excellent agreement. By extrapolation, the thick (blue) point defines μ_X, the intersection point of the extrapolated linear approximation with the line of uniform black strings, $n = n_{st}$.

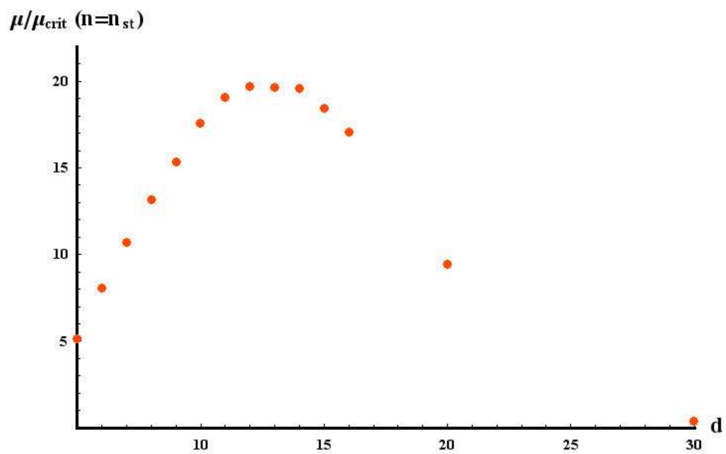

Figure 5: The ratio μ_X/μ_{crit} of extrapolated vs. actual Gregory-Laflamme points for various dimensions. For large d the ratio tends to zero strongly. We interpret that as an indication for a higher (second) order phase transition, and thus as an indication for Sorkin's critical dimension $D^* =$ "13.5". Interestingly, our graph shows a maximum around $12 \leq d \leq 14$ which coincides with D^*.

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A. The Calculation of $\mathbf{Pf}\left(\int (\Phi_{,z})^2 \, d\,V_{d-1}\right)$

We start by replacing Φ by a sum over images

$$\mathrm{Pf}\left(\int (\Phi_{,z})^2 \, d\,V_{d-1}\right) = \tag{A.1}$$
$$= \mathrm{Pf}\left((d-3)^2 \rho_0^{2\,(d-3)} \sum_{(m,n)\in\mathbb{Z}^2} \int_{-\frac{L}{2}}^{\frac{L}{2}} d\,z \int_0^\infty r^{d-3}\, d\,r \frac{(z+m\,L)\;(z+n\,L)}{(r^2+(z+m\,L)^2)^{\frac{d-1}{2}}\;(r^2+(z+n\,L)^2)^{\frac{d-1}{2}}}\right).$$

The summation here goes over all the pairs of mirror images of the black hole, and we transform it as follows

$$\sum_{(m,n)\in\mathbb{Z}^2} \int_{-\frac{L}{2}}^{\frac{L}{2}} d\,z = \sum_{k=m-n\in\mathbb{Z}} \int_{-\infty}^{+\infty} dz = \left[\; |_{k=0} + 2\sum_{k=1}^{\infty}\;\right] \int_{-\infty}^{+\infty} dz \;, \tag{A.2}$$

where according to (4.4,4.6)

$$|_{k=0} \;=\; \left(\Phi_{,z}{}^2\right)_{SI}, \tag{A.3}$$
$$2\sum_{k=1}^{\infty} \;=\; \left(\Phi_{,z}{}^2\right)_{REG}. \tag{A.4}$$

This transformation, replacing the integration over $-\frac{L}{2} < z < \frac{L}{2}$ by integration over the covering space $-\infty < z < \infty$, is justified since the integral is invariant under translations in the direction of the periodic coordinate z, and thus we can replace summation over images with fixed $k \equiv m-n$ by a summation of integrals over stripes $(m-1/2)L \leq z \leq (m+1/2)L$ which altogether exhaust the whole range of z in the covering space: for fixed k we have $\sum_m \int_{-L/2}^{+L/2} dz\, I_{m,n} = \sum_m \int_{(m-1/2)L}^{(m+1/2)L} dz\, I_{k,0}$, where $I_{m,n}$ is the integrand in (A.1). Then we wrote the sum over k as a sum over natural numbers $k \in \mathbb{N}$ (and multiplied by two) when $k\,L$ is the distance of the black hole from its mirror image, and we separated the singular term which corresponds to $k=0$.

Now, we have to regularize just the singular term that comes from the self-interaction

$$\mathrm{Pf}\left(\int \Phi_{,z}{}^2 \, dV_{d-1}\right) = \mathrm{Pf}\left(\int \left(\Phi_{,z}{}^2\right)_{SI} \, dV_{d-1}\right) + \int \left(\Phi_{,z}{}^2\right)_{REG} dV_{d-1}. \tag{A.5}$$

This is done according to the idea of Hadamard's partie finie regularization introduced in section 2. Transforming the singular term to polar coordinates (ρ,χ) and imposing a cut-off at $\rho=\varepsilon$ we see that it is proportional to

$$\int_\varepsilon^\infty \frac{d\rho}{\rho^{d-2}} \propto \frac{1}{\varepsilon^{d-3}} \;,$$

namely

$$\mathrm{Pf}\left(\int \left(\Phi_{,z}{}^2\right)_{SI} \, dV_{d-1}\right) = 0 \;. \tag{A.6}$$

We turn to the regular part

$$\int \left(\Phi_{,z}{}^2\right)_{REG} dV_{d-1} =$$
$$= 2\,\rho_0^{2\,(d-3)}\,(d-3)^2 \sum_{k\in\mathbb{N}} \int_{-\infty}^{\infty} d\,z \int_0^{\infty} r^{d-3}\,d\,r\,\frac{z}{(r^2+z^2)^{\frac{d-1}{2}}} \cdot \frac{(z+k\,L)}{(r^2+(z+k\,L)^2)^{\frac{d-1}{2}}}. \quad \text{(A.7)}$$

Integration by parts gives

$$2\,\rho_0^{2\,(d-3)}\,(d-3) \sum_{k\in\mathbb{N}} \int_{-\infty}^{\infty} d\,z \int_0^{\infty} r^{d-3}\,d\,r\,\frac{1}{(r^2+z^2)^{\frac{d-3}{2}}} \cdot \frac{\partial}{\partial z}\left(\frac{(z+k\,L)}{(r^2+(z+k\,L)^2)^{\frac{d-1}{2}}}\right). \quad \text{(A.8)}$$

This integral can be rewritten in the form

$$-2\,\rho_0^{2\,(d-3)} \sum_{k\in\mathbb{N}} \frac{\partial^2}{\partial b^2}\bigg|_{b=k\,L} \int_{-\infty}^{\infty} d\,z \int_0^{\infty} r^{d-3}\,d\,r\,\frac{1}{(r^2+z^2)^{\frac{d-3}{2}}} \cdot \frac{1}{(r^2+(z+b)^2)^{\frac{d-3}{2}}} =, \quad \text{(A.9)}$$

transforming to polar coordinates and introducing a dimensionless variable $u := \frac{\rho}{b}$ leads us to

$$= -2\,\rho_0^{2\,(d-3)}\Omega_{d-3} \sum_{k\in\mathbb{N}} \frac{\partial^2}{\partial b^2}\bigg|_{b=k\,L} \frac{1}{b^{d-5}} \int_0^{\infty} u\,d\,u \int_0^{\pi} d\,\chi \frac{\sin^{d-3}(\chi)}{(u^2+2\,u\cos(\chi)+1)^{\frac{d-3}{2}}}. \quad \text{(A.10)}$$

The integral over the angle χ gives

$$\int_0^{\pi} d\,\chi \frac{\sin^{d-3}(\chi)}{(u^2+2\,u\cos(\chi)+1)^{\frac{d-3}{2}}} = \begin{cases} \frac{\Omega_{d-2}}{\Omega_{d-3}} & 0<u\le 1 \\ \frac{\Omega_{d-2}}{\Omega_{d-3}} \cdot \frac{1}{u^{d-3}} & 1<u<\infty \end{cases}. \quad \text{(A.11)}$$

Finally, performing the b-derivatives and integrating over u gives us the final result

$$\mathrm{Pf}\left(\int (\Phi_{,z})^2\,d\,V_{d-1}\right) = -\frac{\rho_0^{2\,(d-3)}}{L^{d-3}}\Omega_{d-2}(d-4)(d-3)\zeta(d-3). \quad \text{(A.12)}$$

B. The next to leading correction to the "Archimedes" effect

In [5] we computed the leading order for the inter-polar distance defined to be the proper distance between the "poles" of the black hole measured around the compact dimension

$$L_{\text{poles}} = 2\int_{z_H}^{L/2} dz\,\sqrt{g_{zz}}, \quad \text{(B.1)}$$

where z_H denotes the location of the horizon. It is convenient to define the dimensionless quantities

$$y :\ =\ 1 - \frac{L_{\text{poles}}}{L}\ , \quad \text{(B.2)}$$
$$\eta := \frac{\rho_H}{L}\ , \quad \text{(B.3)}$$

where ρ_H is the location of the horizon in isotropic coordinates, related to the Schwarzschild radius, ρ_0 through

$$ {\rho_0}^{d-3} = 4\,\rho_H^{d-3} \ . \tag{B.4} $$

In [5] we found

$$ y = 2\, I_d\, \eta + o(\eta) \ , \tag{B.5} $$

where the constants I_d are defined by

$$ I_d := 1 - \int_0^1 \left(\left(1 + w^{d-3}\right)^{\frac{2}{d-3}} - 1 \right) \frac{dw}{w^2} = 4^{1/k}\sqrt{\pi}\, \frac{\Gamma\left(\frac{k-1}{k}\right)}{\Gamma\left(\frac{1}{2} - \frac{1}{k}\right)}, \quad k = d-3 \ , \tag{B.6} $$

and are a monotonic function of d: $I_5 = 0$, $I_6 = 0.6845$, $I_\infty = 1$.

Here we shall compute further corrections for y, getting up to order $d-2$ in η. Let us start by identifying the necessary orders at each patch. In order to compute y at order η^s it is necessary to know the asymptotic metric up to order s, but in the near zone it is sufficient to know the metric up to order $s-1$, due to the division by L in the definition (B.2). Thus for the leading result (B.5) it is sufficient to use the asymptotic zone metric up to order 1, which is the same as order 0, namely the flat metric, and in the near zone up to order 0, namely Schwarzschild. Here we will use our information about the metric in both zones up to order $d-3$, which includes the Newtonian approximation in the asymptotic zone, and the first (monopole) correction in the near zone. Since the next order in the asymptotic zone always vanishes, our result will hold up to η^{d-2}. [10]

The way to compute L_{poles} is to pick some mid-point Z, divide the integration between the two zones

$$ \frac{1}{2}\, L_{\rm poles} = \int_{z_H}^{Z} dz \sqrt{g_{zz}^{\rm (near)}} + \int_{Z}^{L/2} dz \sqrt{g_{zz}^{\rm (asymp)}}, \tag{B.7} $$

and confirm that the result is independent of the choice of mid-point.

In the asymptotic zone we have (3.5,3.6)

$$ g_{zz}^{\rm (asymp)} = 1 + \frac{\Phi}{d-3} + \mathcal{O}\left(\rho_H^{2\,(d-3)}\right) \ . \tag{B.8} $$

Thus the contribution to $L_{\rm poles}/(2L)$ (B.7) is

$$ \left(\frac{L_{\rm poles}}{2L}\right)_{\rm asymp} := \frac{1}{L} \int_Z^{L/2} dz \sqrt{g_{zz}^{\rm (asymp)}} = $$
$$ = \frac{1}{L} \int_Z^{\frac{L}{2}} dz \left(1 + \sum_{n=-\infty}^{\infty} \frac{\rho_0^{d-3}}{2(d-3)\ |z + n\,L|^{d-3}} \right) + \mathcal{O}\left(\rho_0^{2\,(d-3)}\right) = $$

[10] In principle, we have enough information to get up to order $2(d-3)-1$, since there are no corrections in the asymptotic zone up to that order and in the near zone there are known corrections from matching with the Newtonian potential (see subsection 3.3). However, we did not perform this computation since in 5d, the only order where we do not know yet the leading behavior since $I_5 = 0$, order $2(d-3)-1 = 3$ coincides with order $d-2$ which we compute here, and thus such a higher order computation would add information only at higher dimensions where it is less needed.

$$= \frac{1}{2} - \frac{Z}{L} + \frac{\rho_0^{d-3}}{2(d-3)(d-4)\,L^{d-3}} \left[\frac{1}{\tilde{z}^{d-4}} + \sum_{n=1}^{\infty} \frac{1}{(n+\tilde{z})^{d-4}} - \sum_{n=1}^{\infty} \frac{1}{(n-\tilde{z})^{d-4}} \right]_{1/2}^{Z/L} + \\ + \mathcal{O}\left(\rho_0^{2\,(d-3)}, \right) \tag{B.9}$$

where in the last equality we defined $\tilde{z} := z/L$. For $\tilde{z} = 1/2$ the expression in brackets in the last line vanishes. At the other boundary, $\tilde{z} = Z/L$, we should expand up to order $\tilde{z}$ (since we need terms up to order $L^{-(d-2)}$ in the overlap region. Note that here is quite easy to extend the result to order $2(d-3)-1$, which we discussed above, simply by expanding further). Altogether we find

$$\left(\frac{L_{\rm poles}}{2L} \right)_{\rm asymp} = \frac{1}{2} - \frac{Z}{L} + \frac{\rho_0^{d-3}}{2(d-3)(d-4)\,L^{d-3}} \left[\left(\frac{L}{Z} \right)^{d-4} - 2(d-4)\,\zeta(d-3)\,\frac{Z}{L} \right]. \tag{B.10}$$

We now turn to the near zone. Here at order $(d-3)$ we consider only the monopole correction, which leaves us with the Schwarzschild metric, with adjusted parameters. For matching purposes it is convenient to write the Schwarzschild metric in isotropic coordinates

$$ds^2 = -\left(\frac{1-\psi}{1+\psi} \right)^2 dt^2 + (1+\psi)^{\frac{4}{d-3}} \left(d\rho_c^2 + \rho_c^2\, d\Omega_{d-2}^2 \right), \tag{B.11}$$

where

$$\psi = \left(\frac{\rho_H}{\rho_c} \right)^{d-3} . \tag{B.12}$$

To incorporate the correction we could have used (3.18) in a straightforward manner, but here we take an alternative route. In order to match we are free to adjust ρ_H, rescale t and re-parametrize ρ_c. Since we wish to retain the isotropic form we consider only a rescaling of ρ_c. Matching of g_{zz} is achieved by the transformation

$$\begin{aligned} \rho_c &\to (1+\delta)\,\rho_c \ , \\ \delta &:= \frac{1}{2(d-3)}\,\Phi_0 = \frac{\zeta(d-3)}{d-3} \left(\frac{\rho_0}{L} \right)^{d-3} \ , \end{aligned} \tag{B.13}$$

where Φ_0 is the Newtonian potential at the origin due to the images. Matching of g_{tt} requires also

$$t \to (1 + (d-3)\,\delta)\,t \ . \tag{B.14}$$

One is free to change ρ_H as well – this is a change of scheme, or reparametrization of the branch of solutions. Here it is convenient to choose ρ_H to remain unchanged

$$\delta\rho_H = 0 \ , \tag{B.15}$$

(compare this scheme with the scheme of (3.18) which guarantees zero matching at PN order and in the current context would be expressed as $\rho_H \to \rho_H\,(1+\delta)$.)

We may now calculate the contribution of the near zone to $L_{\text{poles}}/(2L)$ (B.7)

$$\begin{aligned}
&\left(\frac{L_{\text{poles}}}{2L}\right)_{\text{near}} := \frac{1}{L}\int_{z_H}^{Z} dz\,\sqrt{g_{zz}^{(\text{near})}} = \\
&= \frac{1}{L}\int_{z_H}^{Z} dz\,(1+\psi)^{2/(d-3)} = \\
&= \frac{\rho_H}{L}\int_{1/(1+\delta)}^{Z/\rho_H}\left(1+\left(\frac{\rho_H}{z\,(1+\delta)}\right)^{d-3}\right)^{2/(d-3)}(1+\delta)\frac{dz}{\rho_H} = \\
&= \frac{\rho_H}{L}\int_{1}^{Z(1+\delta)/\rho_H}\left(1+\left(\frac{\rho_H}{z}\right)^{d-3}\right)^{2/(d-3)}\frac{dz}{\rho_H} = \\
&= \frac{\rho_H}{L}\left[-I_d+\frac{Z\,(1+\delta)}{\rho_H}-\int_0^{\rho_H/(Z(1+\delta))}\left((1+t^{d-3})^{2/(d-3)}-1\right)\frac{dt}{t^2}\right],
\end{aligned} \tag{B.16}$$

where in passing from the second to the third lines we used the adjustments (B.13), including the adjustment in the location of the horizon induced by the rescaling of ρ_c; in the next line we performed a change of variables $(1+\delta)\,z \to z$ and in passing to the fifth we changed to $t := \rho_H/z$. Finally expanding the integral up to order ρ_H^{d-3} we find the total contribution from the near zone to be

$$\left(\frac{L_{\text{poles}}}{2L}\right)_{\text{near}} = \frac{\rho_H}{L}\left[-I_d+\frac{Z\,(1+\delta)}{\rho_H}-\frac{2}{(d-3)(d-4)}\left(\frac{\rho_H}{Z\,(1+\delta)}\right)^{d-4}\right]. \tag{B.17}$$

Adding up (B.10) and (B.17) we find that indeed all the Z dependence disappears to the prescribed order once we recall the definition of δ (B.13) and the relation between ρ_H, ρ_0 (B.4), and we are left with the final answer

$$\frac{L_{\text{poles}}}{2L} = \frac{1}{2} - \eta\, I_d + o\left(\left(\frac{\rho_0}{L}\right)^{d-2}\right), \tag{B.18}$$

or equivalently

$$\boxed{y = 2\,I_d\,\eta + o\left(\left(\tfrac{\rho_0}{L}\right)^{d-2}\right).} \tag{B.19}$$

Summarizing, our result is expressed in terms of y, η, I_d whose definitions are given in (B.2,B.3,B.6). It turns out that using the scheme choice (B.15) our higher order result (B.19) is precisely our low order result (B.5). In particular we find that in 5d y vanishes up to order $d-2=3$ in η (this is scheme independent) which is consistent with 5d numerical simulations [25] where y appears to be $\mathcal{O}(\eta^4)$ (however, the result in [7] is different).

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